Extending computational complexity theory to include thermodynamic resource costs

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with

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A continuous-time Markov chain sending p(0) to $p(1) = \sum_j P(i \mid j)\, p_j(0)$:

$$\frac{dp_i(t)}{dt} = \sum_j K_{ij}(t) p_j(t)$$

- Example: Dynamics of a digital gate in a circuit
- Example: Dynamics of an entire digital circuit
- Example: Dynamics of a deterministic finite automaton (DFA)
- Example: Dynamics of a Turing Machine (TM)

A continuous-time Markov chain sending p(0) to p(1) = $\sum_j P(i \mid j)\, p_j(0)$:

$$\frac{dp_i(t)}{dt} = \sum_j K_{ij}(t) p_j(t)$$

$$\frac{dS(p(t))}{dt} = \dot{Q}(t) + \dot{\Sigma}(t)$$

- $\dot{Q}(t) = \sum_{ij} K_{ij}(t) p_j(t) \ln \frac{K_{ji}(t)}{K_{ij}(t)}$ *Entropy flow rate*
- $\dot{\Sigma}(t) = \sum_{ij} K_{ij}(t) p_j(t) \ln \frac{K_{ij}(t) p_j(t)}{K_{ji}(t) p_i(t)}$ *Entropy production rate*

- ***Entropy production (EP) rate is non-negative***

Van denBroeck and Esposito, *Physica A*, 2015

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$$\frac{dS(p(t))}{dt} = \dot{Q}(t) + \dot{\Sigma}(t)$$

Integrate over time: $-\Delta Q = \Delta\Sigma - \Delta S$

- $\Delta S = S(p_1) - S(p_0)$ is gain in Shannon entropy of p
- $-\Delta Q$ is (Shannon) entropy flow from system between t = 0 and t = 1
- $\Delta\Sigma$ is total entropy production in system between t = 0 and t = 1
  - ***cannot be negative***

  (I.e., the second law of thermodynamics)

Van denBroeck and Esposito, *Physica A*, 2015

For many *non-Markvonian* chains sending p(0) to p(1) = $\sum_j P(i \mid j)\, p_j(0)$:

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  - ***cannot be negative***
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Ptaszynski and Esposito, *PRL*, 2019

GENERALIZED LANDAUER BOUND

- System connected to multiple reservoirs, e.g., heat baths at different temperatures. (So "kBT " not defined.)
- Arbitrary number of states
- Arbitrary initial distribution p_0
- Arbitrary dynamics $P(x_1 \mid x_0)$

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$$-\Delta Q = \Delta\Sigma + S(p_0) - S(p_1)$$

Entropy Production ($\Delta\Sigma$) is non-negative. So:

"*Generalized Landauer's bound*"

$$-\Delta Q \geq S(p0) - S(p1)$$

BEYOND GENERALIZED LANDAUER

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- Yes.

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Entropy Production ($\Delta\Sigma$) is non-negative.

Are there broadly applicable non-negative lower bounds on $\Delta\Sigma$,

to add to the lower bound $-\Delta Q \geq S(p0) - S(p1)$?

- Yes.
  - Focus on two: ***Speed limit theorem*** (SLT) and ***Mismatch cost***

Use them to investigate
the (thermodynamic) resource costs of
computational machines

BOOLEAN CIRCUITS

- Currently, all mass-produced computers are implemented with circuits.

- The simplest circuit is one without loops or branches (a "straight-line program")

- If set of allowed gates are a universal basis (e.g., NAND gates), then can build a circuit with them to implement any desired Boolean function.

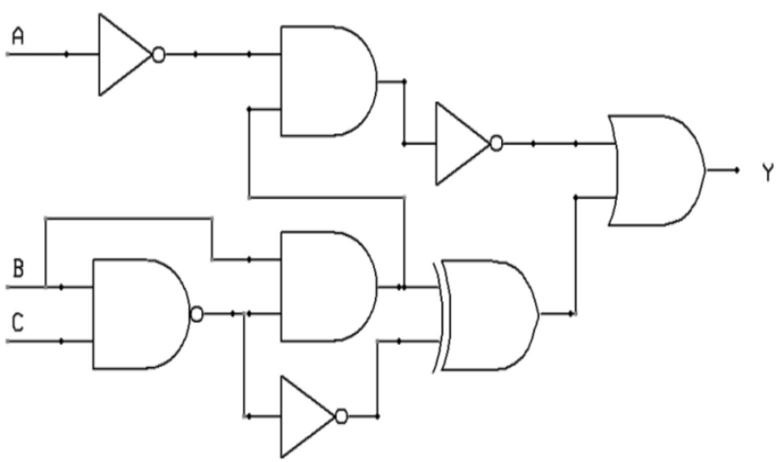
$$-\Delta Q = \Delta\Sigma + S(p_0) - S(p_1)$$

- For fixed $P(x_1 \mid x_0)$, changing p_0 changes Landauer cost, S(p0) − S(p1)

- N.b., the same $P(x_1 \mid x_0)$ - e.g., same AND gate - has different p0, depending on where it is in a circuit.

- So *even for a thermo. reversible gate* ($\Delta\sum$(p0) = *0*), ***changing the gate's location in a circuit*** (changes S(p0) − S(p1) and so) ***changes -ΔQ(p0)***

A	B	C	Y
0	0	0	
0	0	1	
0	1	0	
0	1	1	
1	0	0	
1	0	1	
1	1	0	
1	1	1	



- Changing a gate's location in a circuit changes S(p0) − S(p1), and so changes the heat it produces, -ΔQ(p0)

- Sum those heats over all gates to get minimal heat flow of that circuit

Different circuits implementing *same* Boolean function on *same* input distribution have *different* minimal heat

- Formally, those differences in minimal heat of the circuits are differences in EPs of the circuits, arising due to modularity of gates

➢ A new circuit design optimization problem

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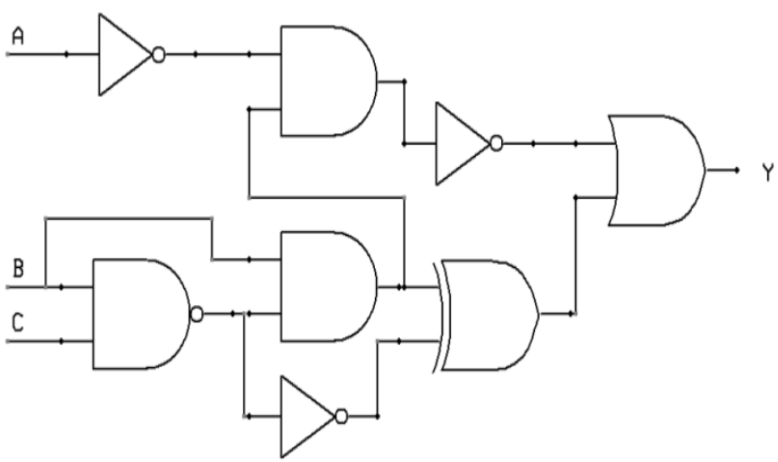

Demaine, E., et al., *Comm. ACM*, 2016

- Considers a similar problem - but incorrectly sets Landauer cost at each gate to same value, KT ln(2).

Original speed limit theorem (SLT): $\Delta\Sigma \geq \frac{L(p(0), p(1))}{A_{0,1}}$

- $L(p(0), p(1))$: L_1 distance from distribution p(0) to distribution p(1)
- $A_{0,1}$: total number of (stochastic) state jumps from t = 0 to t = 1

Since introduced, SLT has been strengthened several ways

(more complicated formulas).

Shiraishi, N., Funo, K.; Saito, K., *PRL* (2018)
Delvenne, J., Falasco, G.; arXiv:2110.13050
Lee, J., et al.; *PRL* (2022)
Van Vu, T., Saito, K.; *PRL* (2023)

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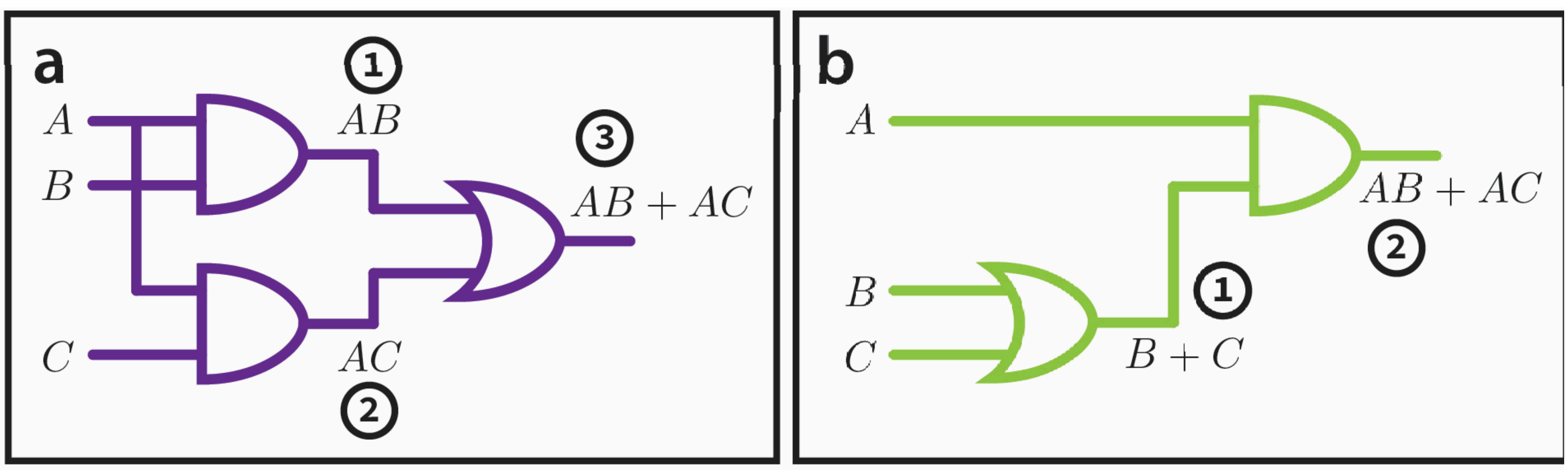
- Suppose uniform initial distribution over all gates and input bits;

•How does the (Lee et al.) SLT bound vary with error rate of gates, for two logically equivalent circuits?



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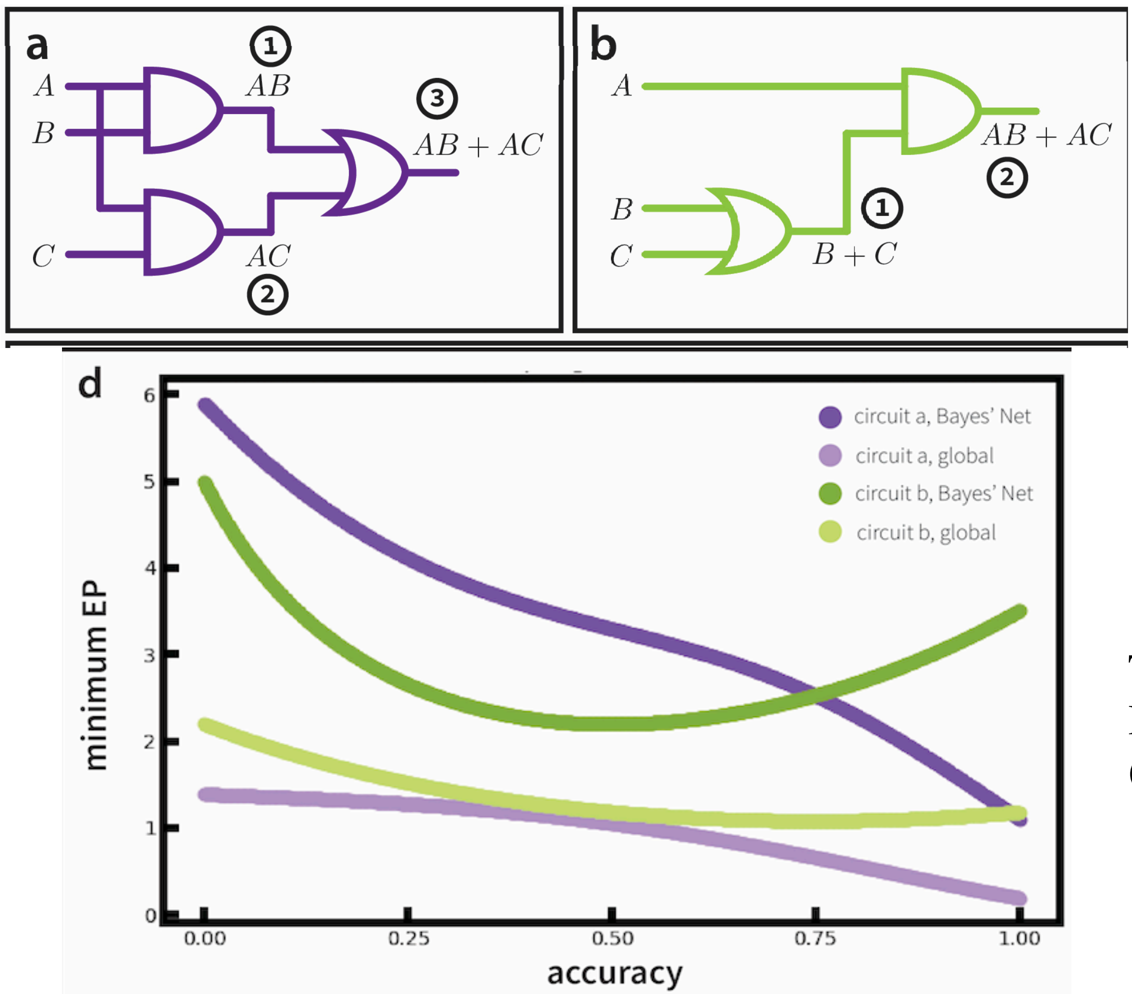

Tasnim, F., Wolpert, D., Korbel J., Lynn, C., et al. (2023)

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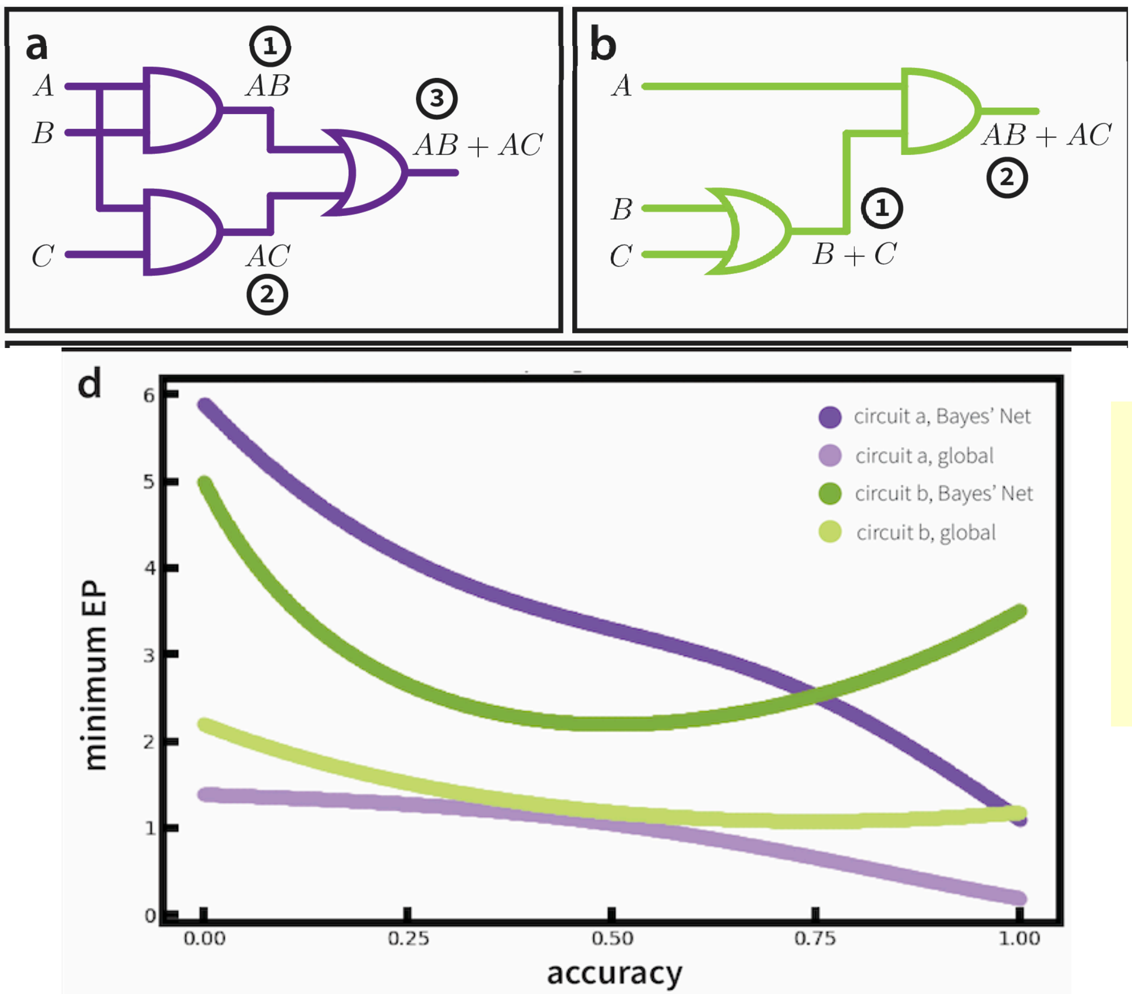

What causes the curves to have these shapes? What are curves for other circuits?

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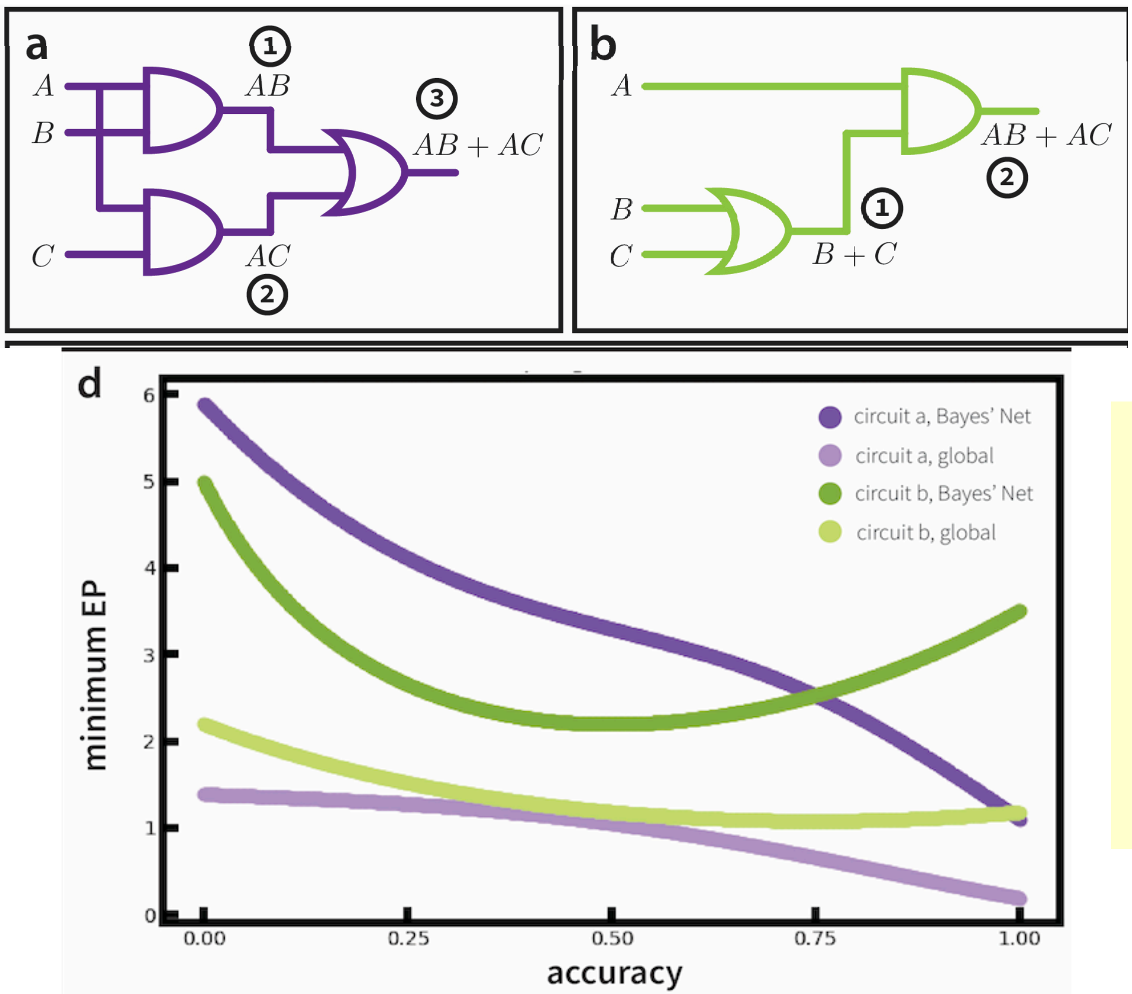

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A: Who knows!

DEPENDENCE OF EP ON INITIAL DISTRIBUTION

- Arbitrary dynamics $P(x_1 \mid x_0)$
- ***Assume system is thermo. reversible for initial distribution q_0***

I.e., $\Delta\Sigma(q0) = 0$

- Run that system with initial distribution $p_0 \neq q0$ instead:

$$\Delta\Sigma(p0) = D(p0 \,||\, q0) - D(p1 \,||\, q1) \geq 0$$

where D(. || .) is relative entropy (KL divergence)

Wolpert, D., Kolchinsky, A., *New J. Phys.* (2020)
Riechers, P.. Gu, M., *Phys. Rev. E* (2021)
Kolchinsky, A., Wolpert D., *arxiv:2103.05734*

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Any nontrivial process that is thermodynamically reversible for one initial distribution ***will be costly*** for any other initial distribution

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$D(p0 \,||\, q0) - D(p1 \,||\, q1)$ is called ***mismatch cost***

Mismatch cost example/

- ***Two*** distinct bit-erasing gates, each with thermo. rev. initial distribution q_0
- Run gates in parallel, on bits x^A and x^B, with initial distribution $p_0(x^A, x^B)$
- Assume $p_0(x^A) = q_0(x^A)$ and $p_0(x^B) = q_0(x^B)$.

- So each gate, by itself, generates zero EP. But:

If $p_0(x^A, x^B)$ statistically couples the bits, then
full system is **not** *thermo. reversible,*
and generates nonzero EP

- **Formally**: Since gates are distinct, the thermo. rev. *joint* distribution is $q_0(x^A, x^B) = q_0(x^A)q_0(x^B)$.

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- Intuition: **Running two thermo. reversible gates in parallel loses information in their initial coupling, and so is not thermo. reversible**.

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- **Broader lesson**: ***Modularity increases EP***

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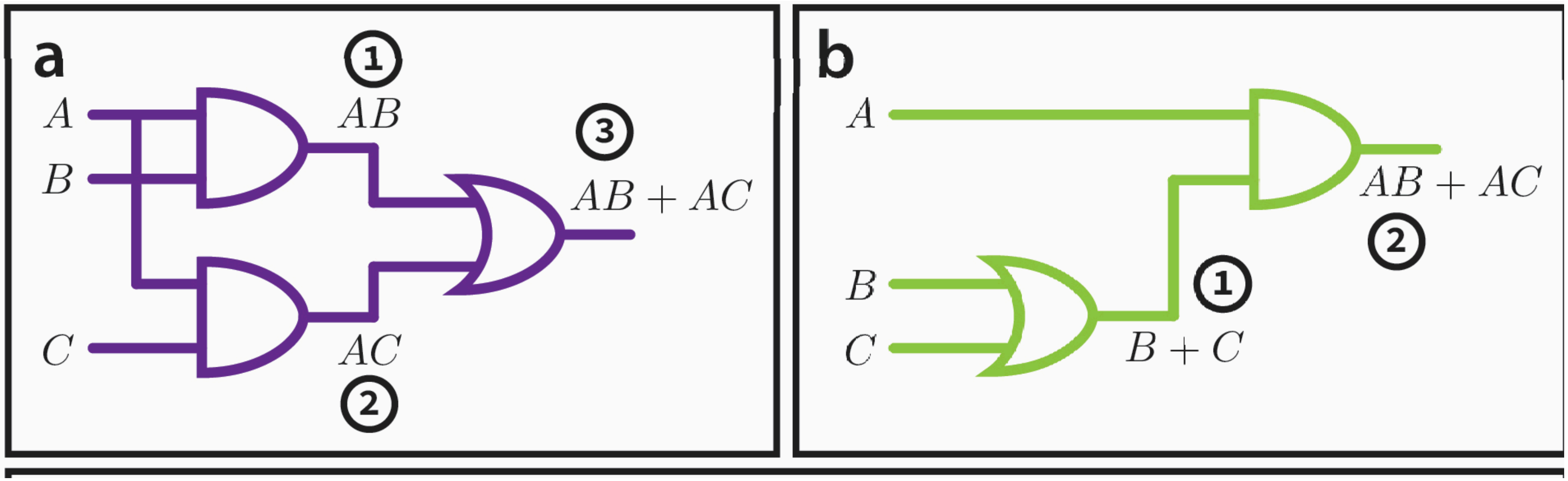
- **Broader lesson**: Whatever its practical benefits might be,

modularity is thermodynamically costly (!)

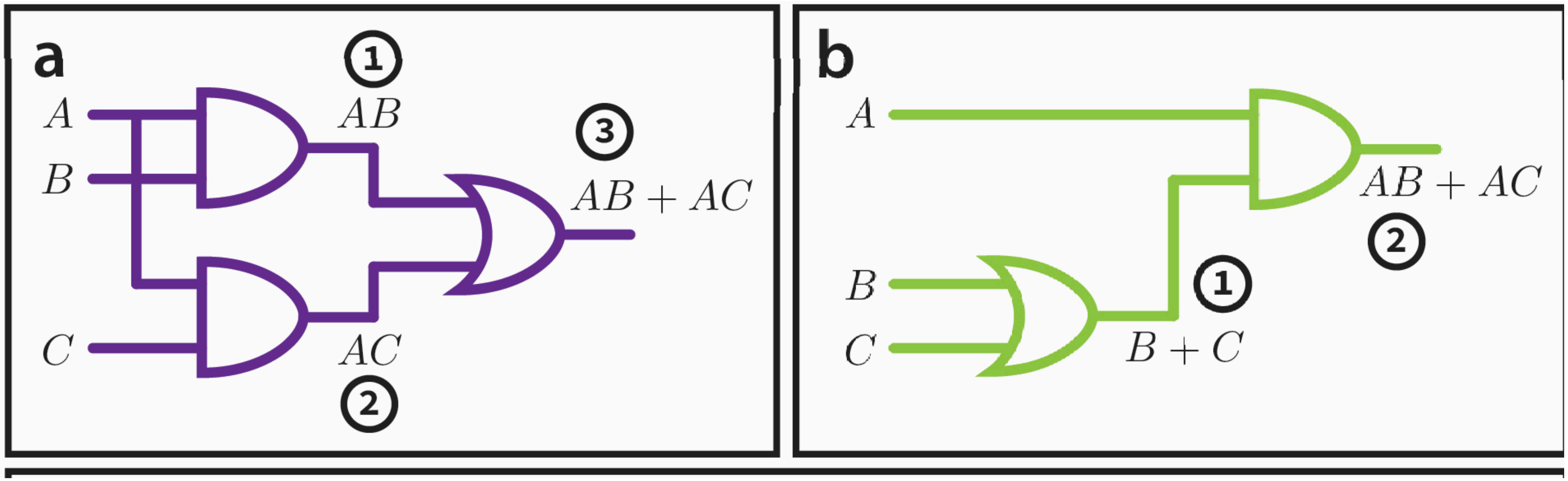
MISMATCH COST OF BOOLEAN CIRCUITS



- Physical process updating each gate in a real circuit depends only on that gate's inputs – it is independent of all other gates.
- Similar to parallel bit erasure.

Wolpert, D., *PRL* (2020)

MISMATCH COST OF BOOLEAN CIRCUITS



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- Assume priors of gates are also all uniformly random

- Then mismatch cost = 0 - for the first use of the circuit

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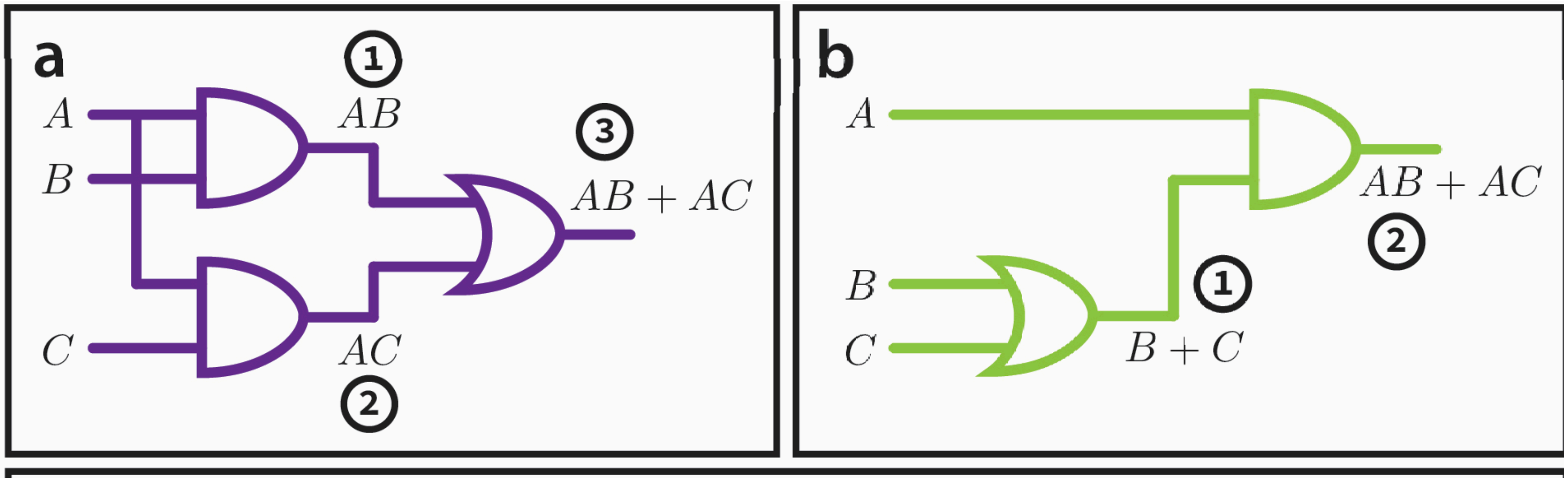

- On first use of circuit, inputs and all gates uniformly random
- Assume priors of gates are also all uniformly random
- Suppose on second use, inputs are again uniformly random – but gates are reinitialized, e.g., to uniformly random.

- Then mismatch cost = 0 - for the second use of the circuit.

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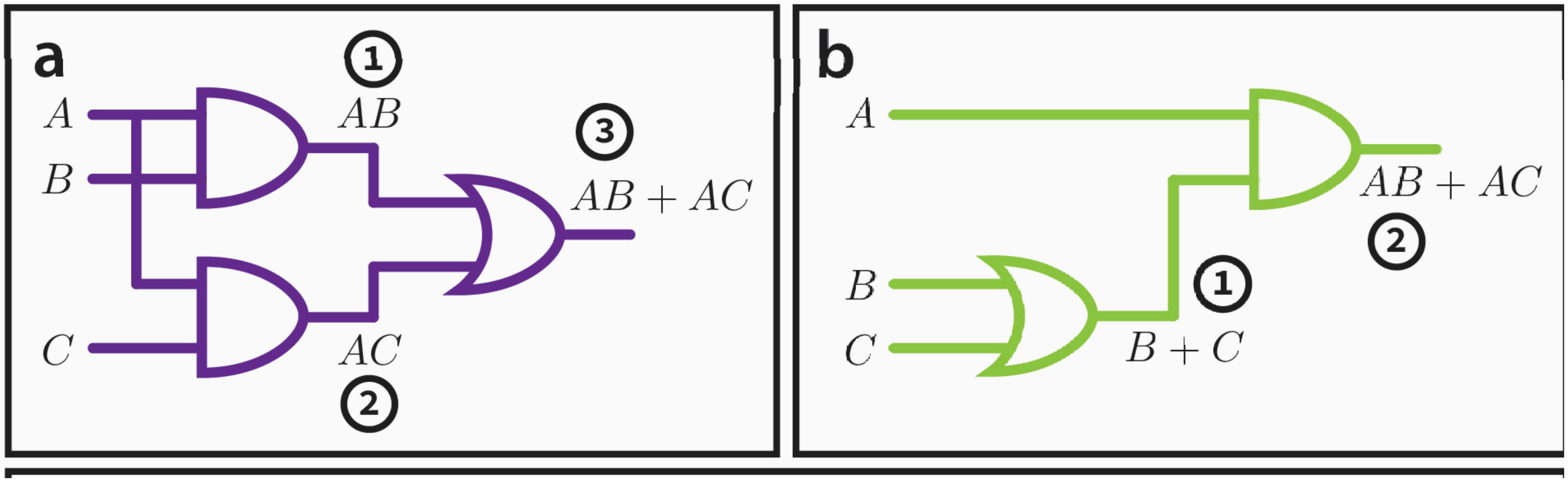

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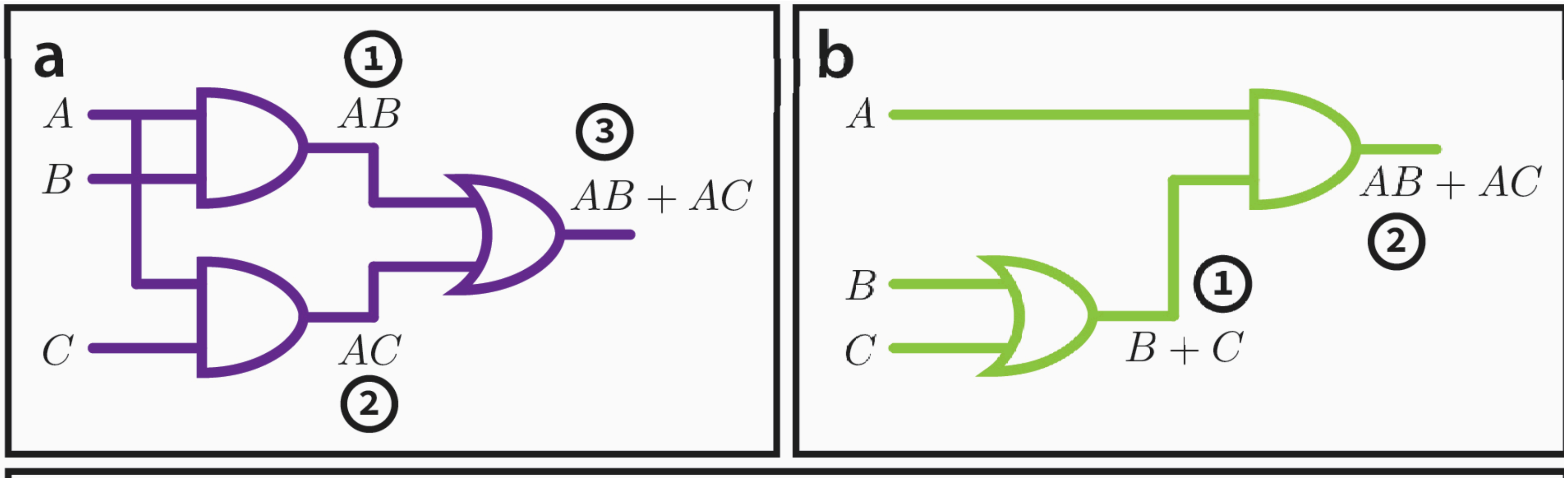

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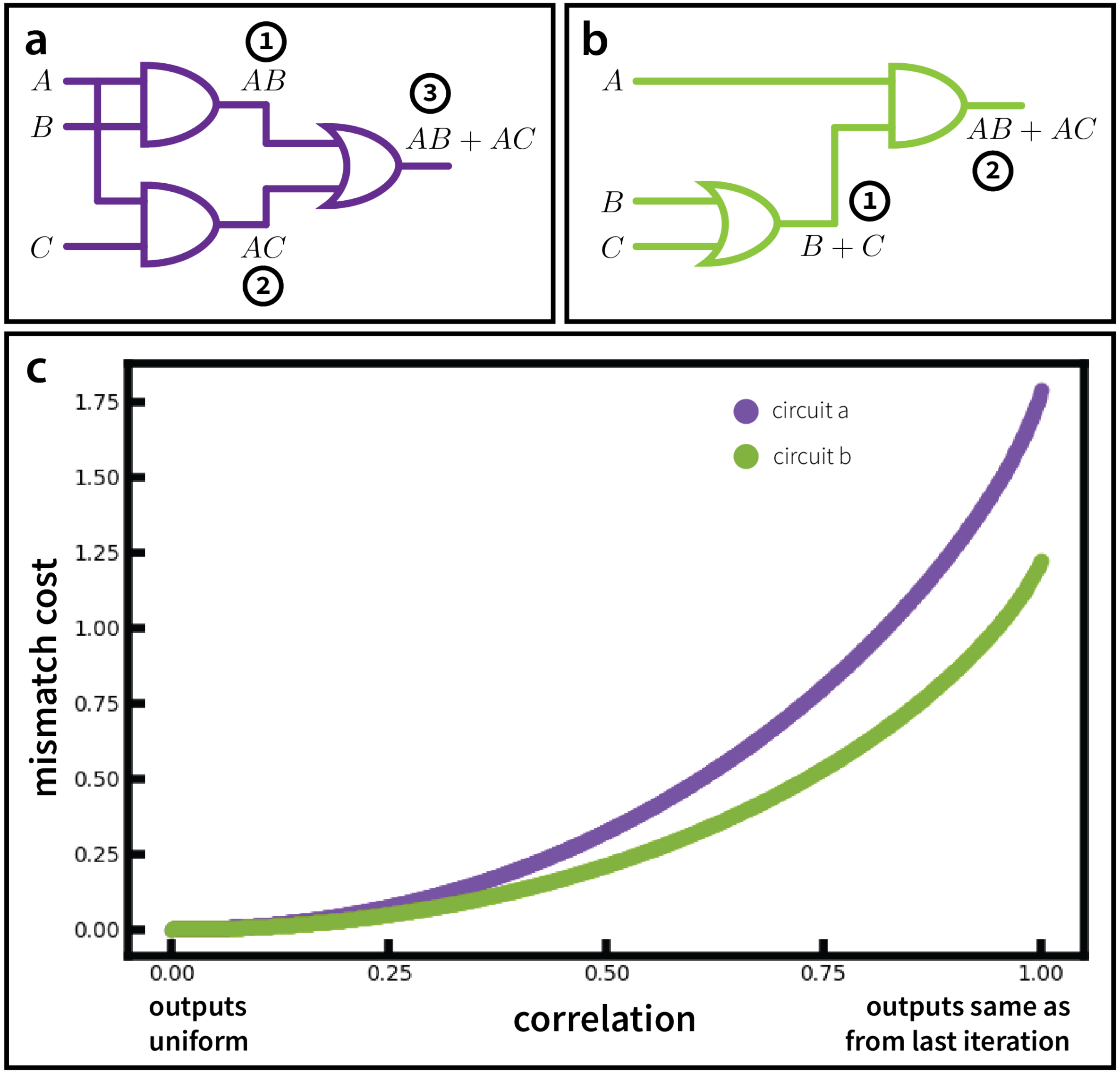
- **Then mismatch cost $\neq$ 0** - for the second use of the circuit.

MISMATCH COST OF BOOLEAN CIRCUITS



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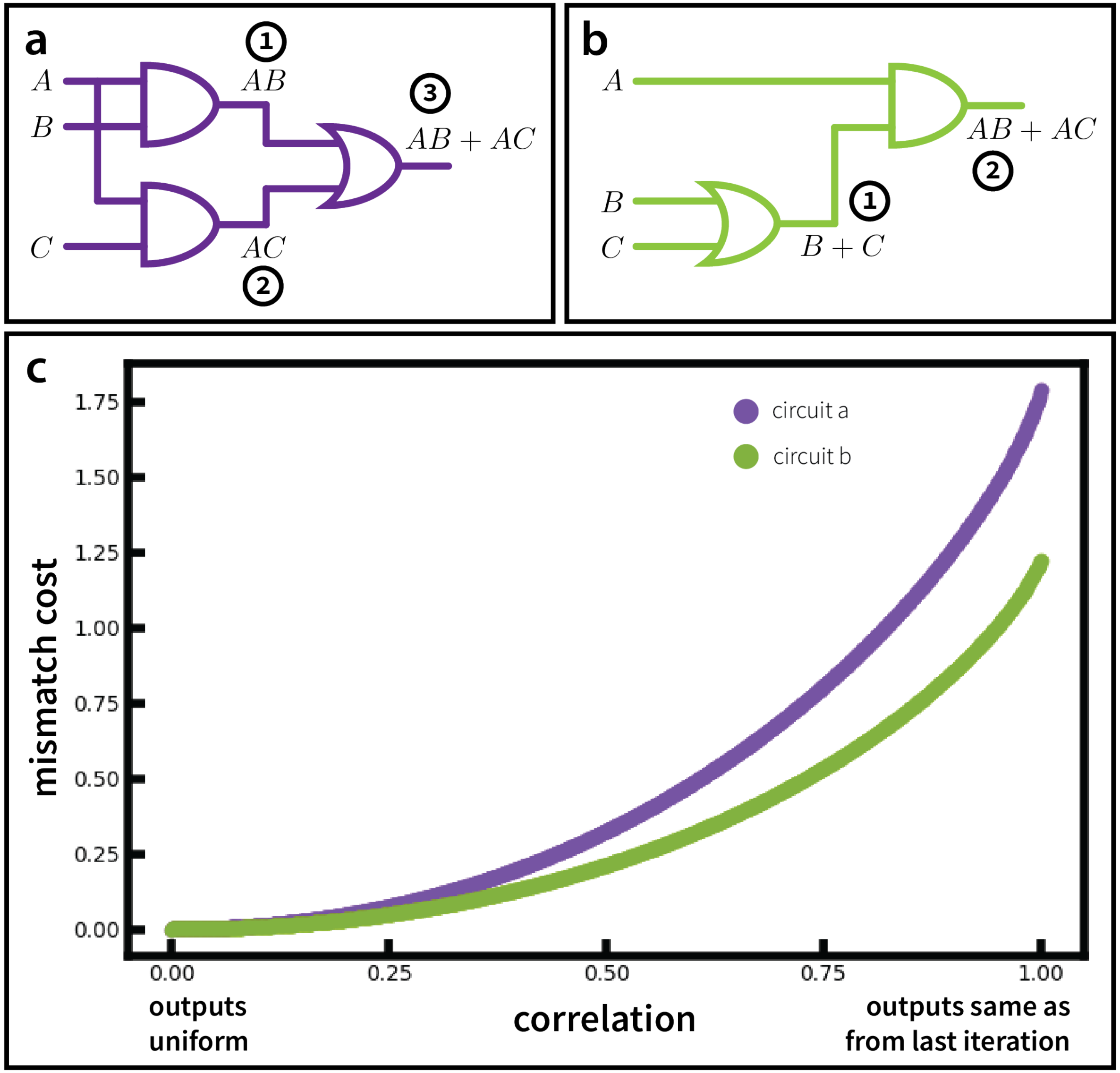

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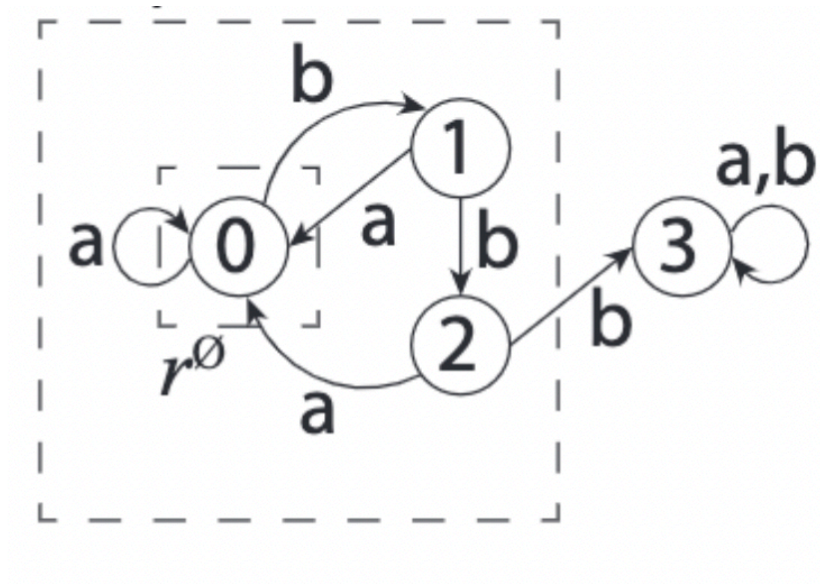
DETERMINISTIC FINITE AUTOMATA (DFA)

- **Simplest computational machine in Chomsky hierarchy**
  - Finite number of states; one initial state, multiple "accept states"
  - Feed in a finite string of bits;
  - Each (bit, state) pair maps to a new state, after which next bit is read
  - A DFA "accepts" a string if it causes the DFA to end in an accept state
  - "Language" of a DFA is all input strings that it accepts
  - Many languages that are not accepted by *any* DFA

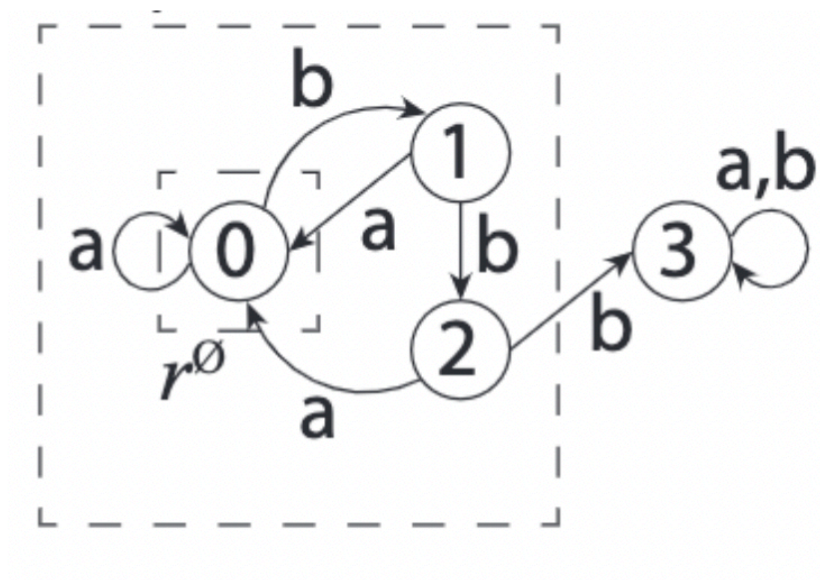
- **Example**: DFA that accepts any string with no more than two successive 'b' bits:



- *Every* digital computer is "local"
  - the only part of memory any processing unit is directly physically coupled to is its current input

- E.g., in a DFA, state update only physically coupled to current input symbol, not any earlier / later symbols

- Results in "modularity (mismatch) cost" – just like parallel bit erasure

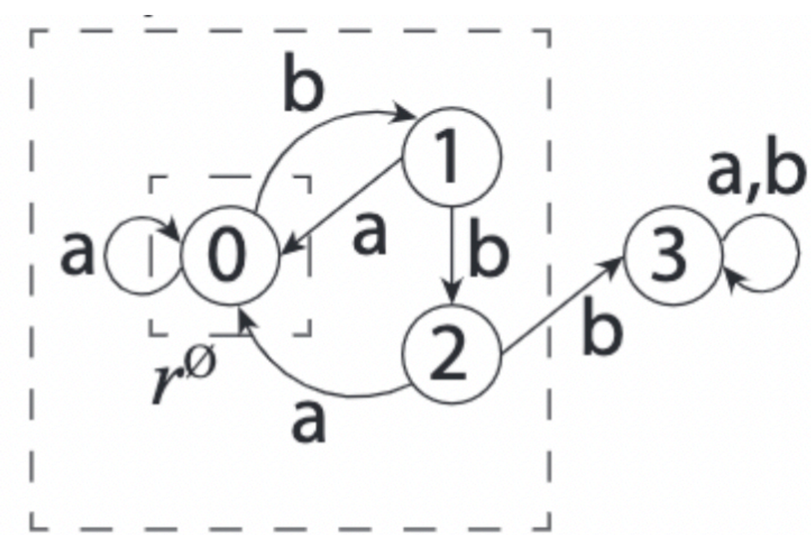


- *Every* (synchronous) digital computer is "periodic"
  - every successive iteration is the same physical process, and so in particular has the same prior.

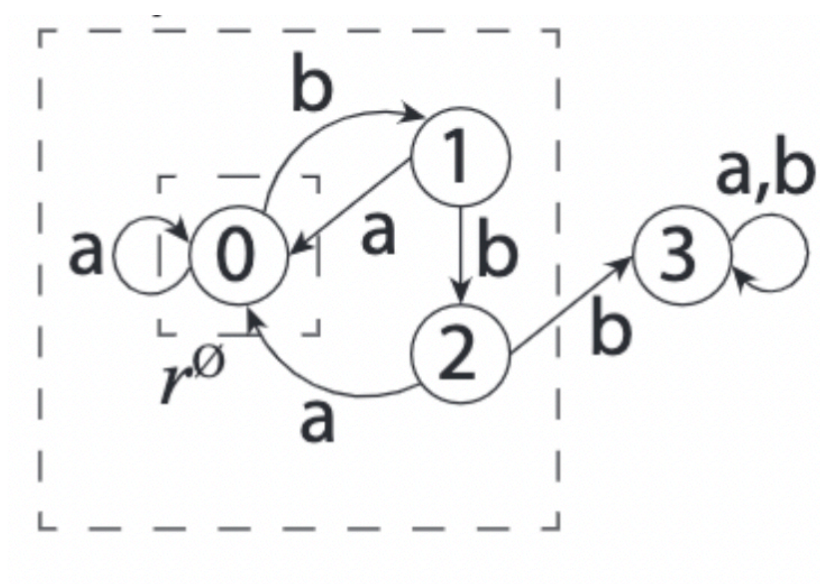
- E.g., in a DFA, every iteration has same prior

- So if prior = actual distribution for iteration i (so zero mismatch cost), *they will differ for iteration i + 1 in general* (so nonzero mismatch cost!)

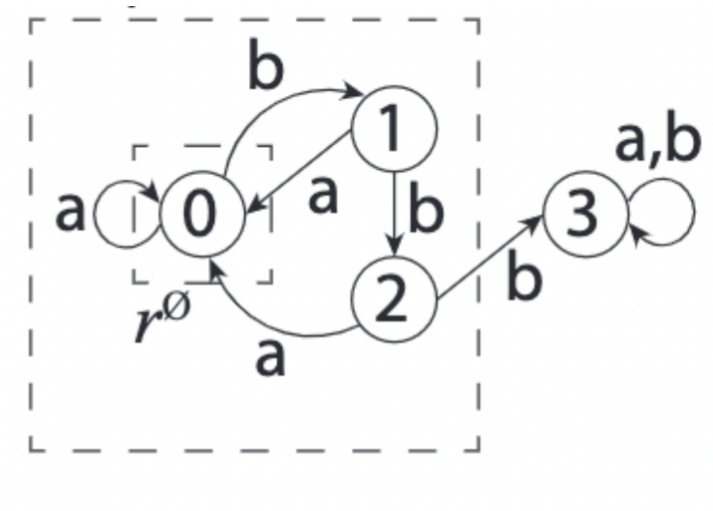
- Results in "modularity (mismatch) cost" – just like parallel bit erasure

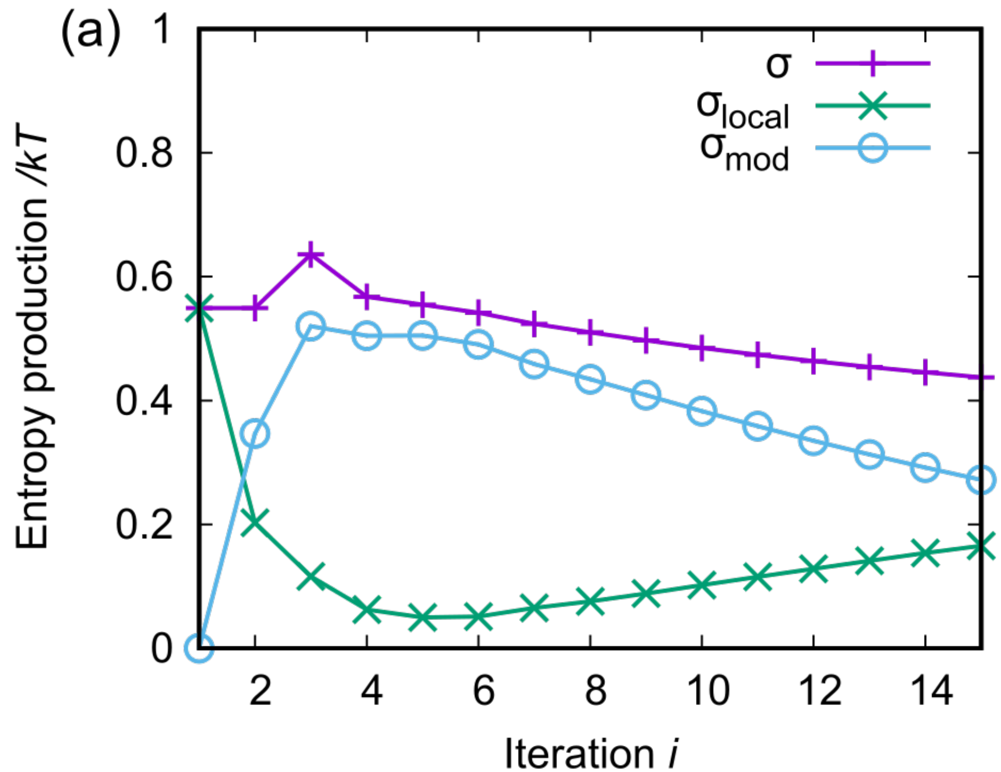


- Total mismatch cost = modularity cost + local cost



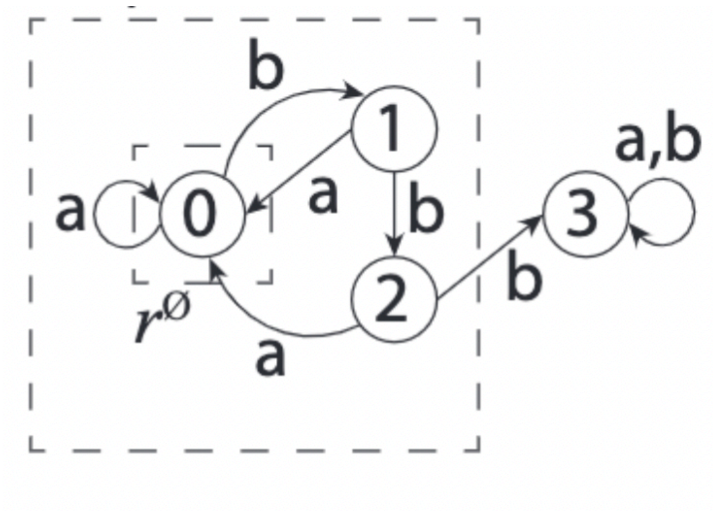
EXAMPLE

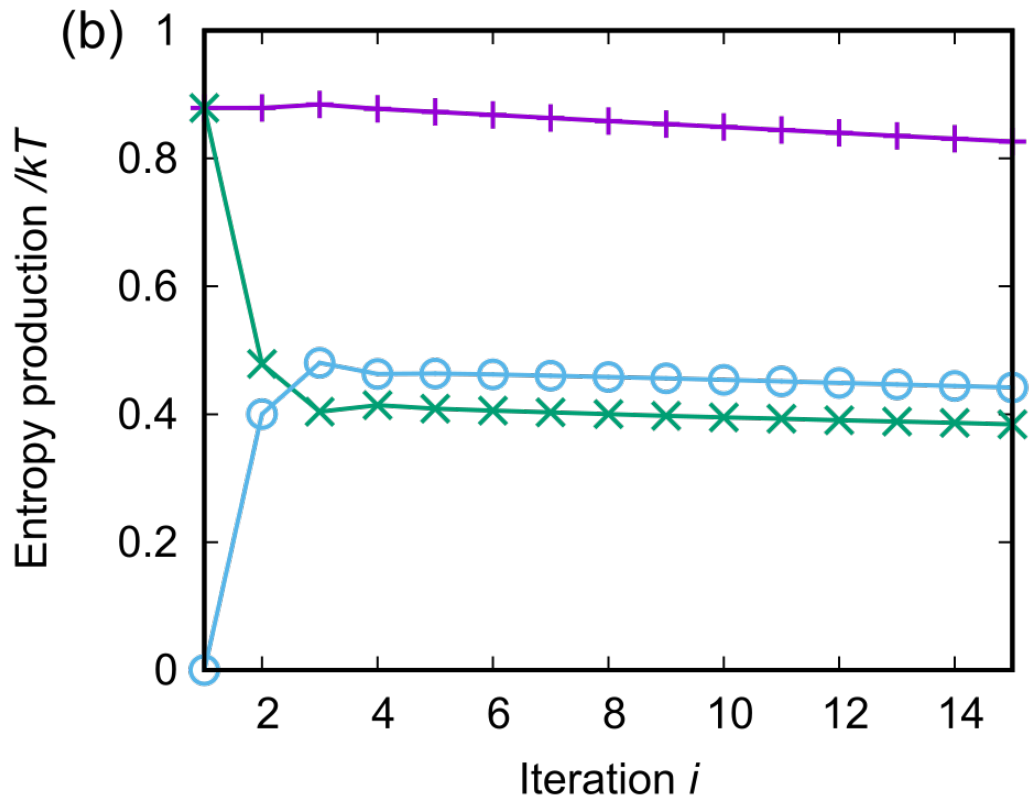




Input strings have IID symbols with equal probability of a and b

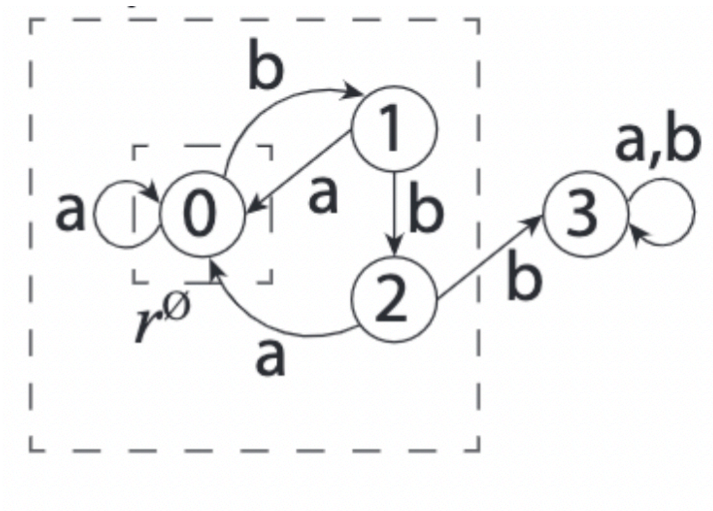
EXAMPLE



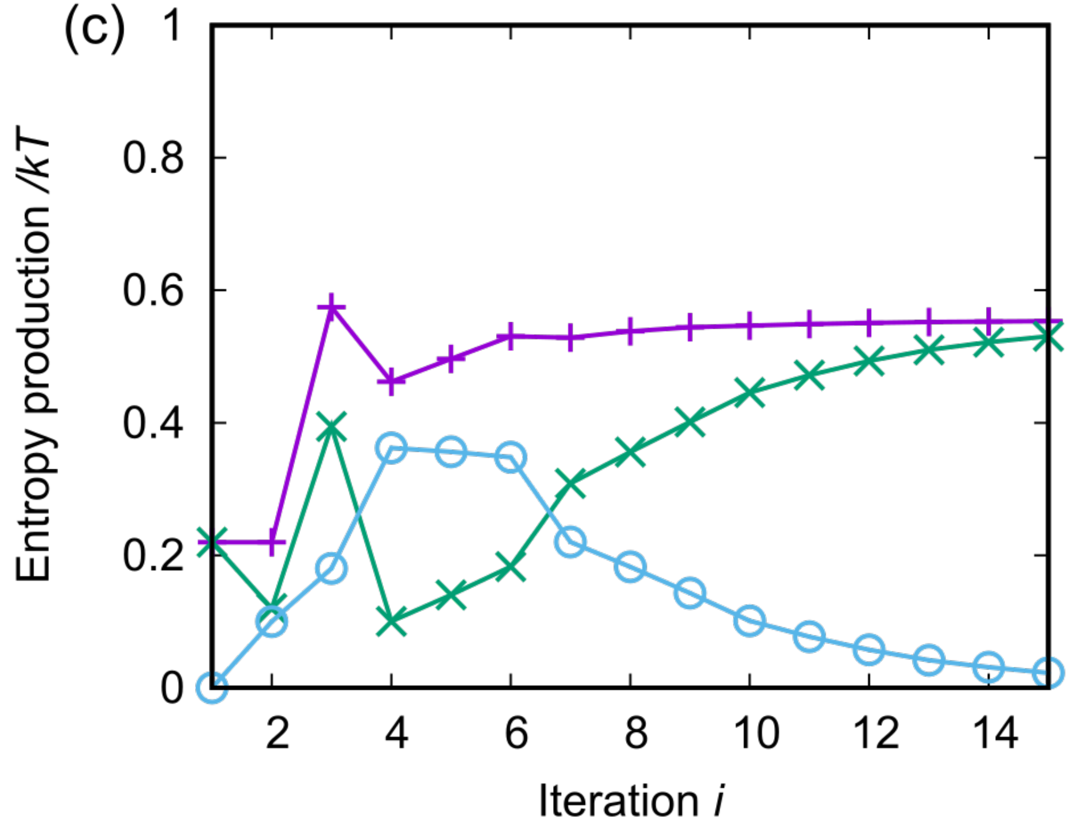


Input strings have IID symbols with probability of a = 0.8

EXAMPLE

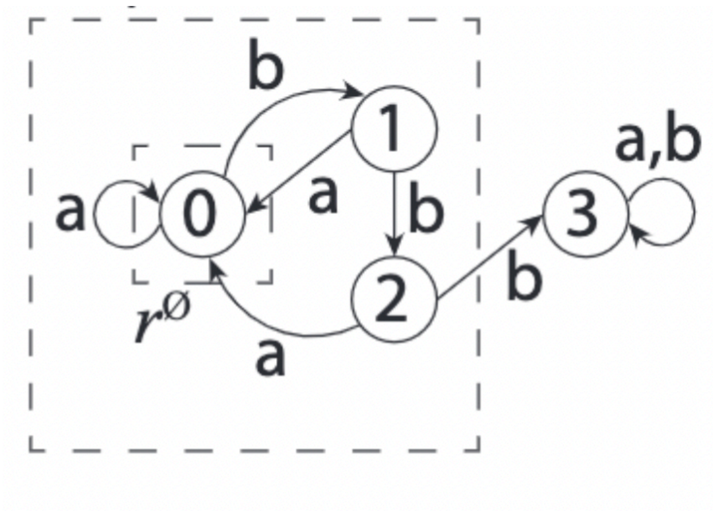


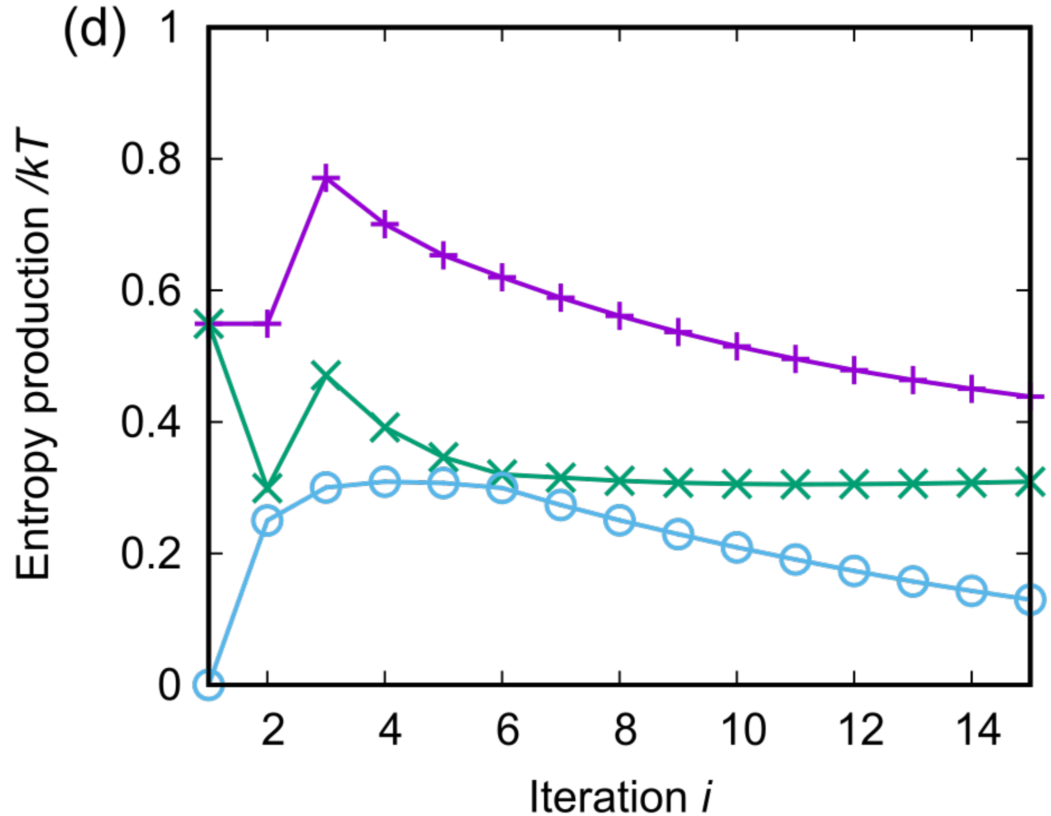
(c)



Input strings are first order Markov chains (starting from uniform probability)

EXAMPLE





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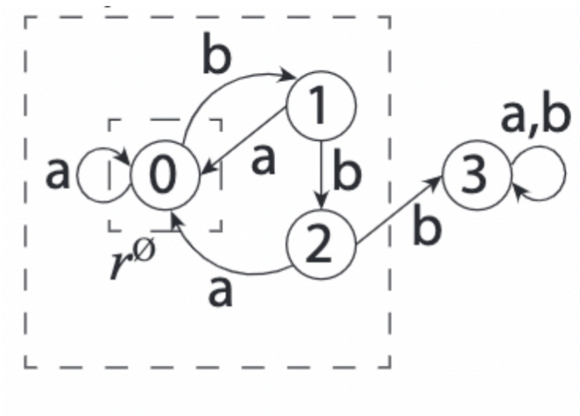
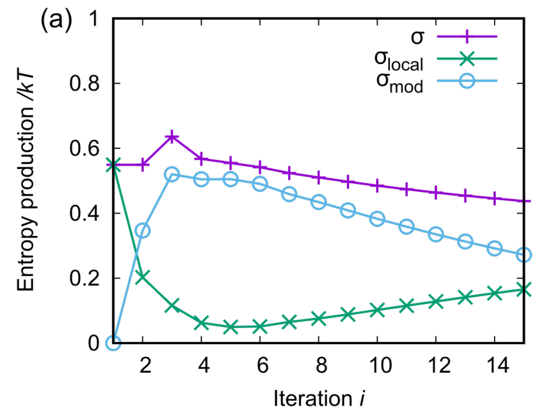
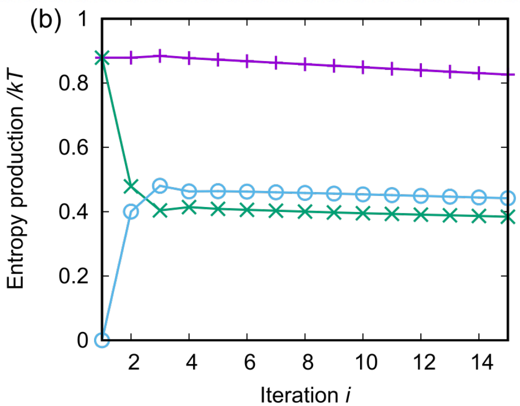
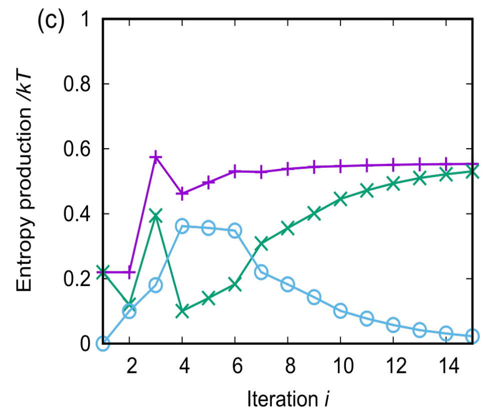
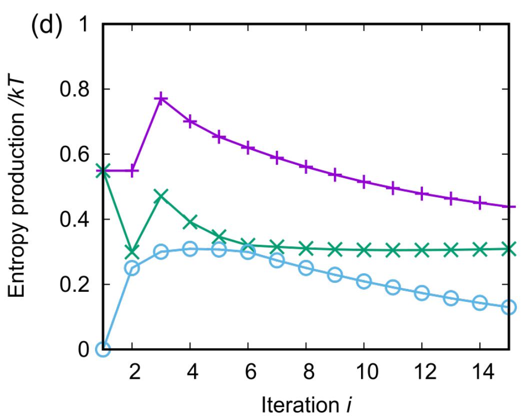




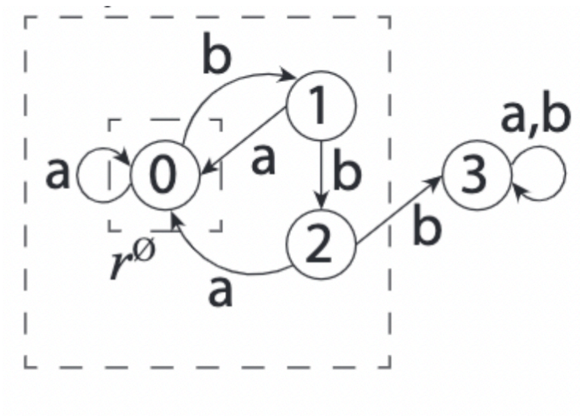


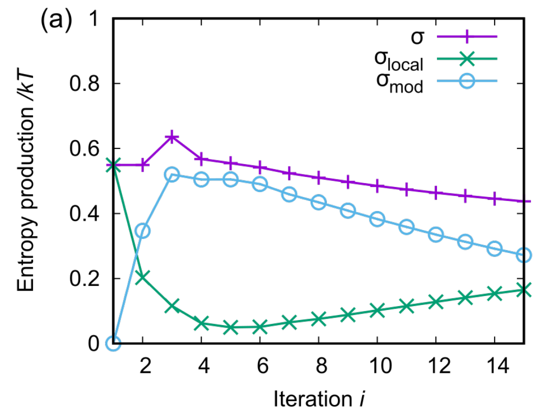


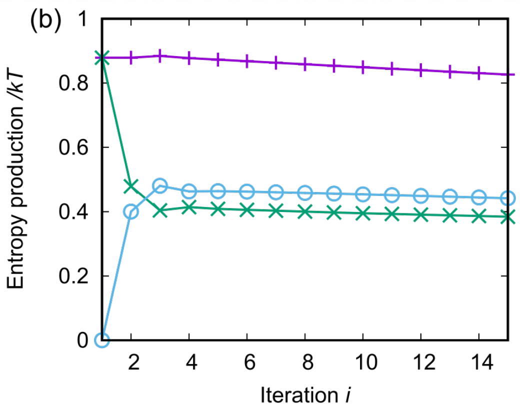


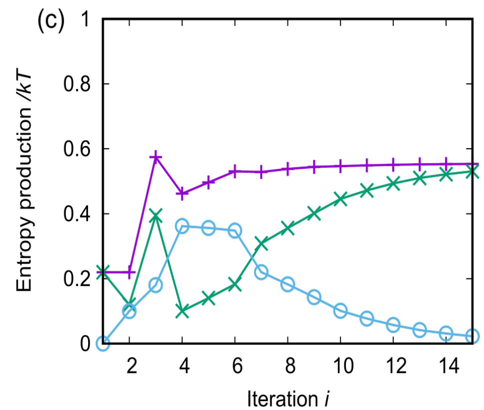
Ouldridge, T., Wolpert, D., arxiv:2208.06895 (2022)

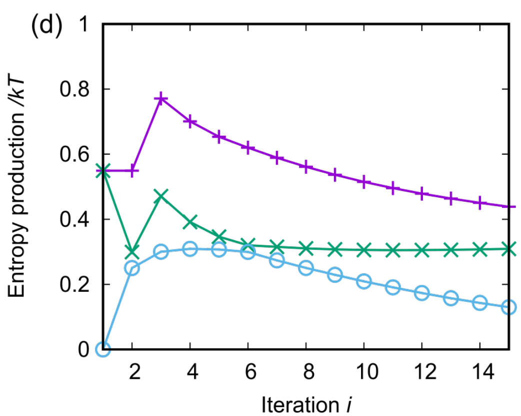
EXAMPLE











What causes curves to have these shapes?
What are curves for other DFAs?

A: Who knows!

All these bounds on thermodynamic cost of computers hold *independent of nitty gritty details of physical system implementing the computer*

Other recent results based on “nitty gritty details”of computers implemented using CMOS technology.

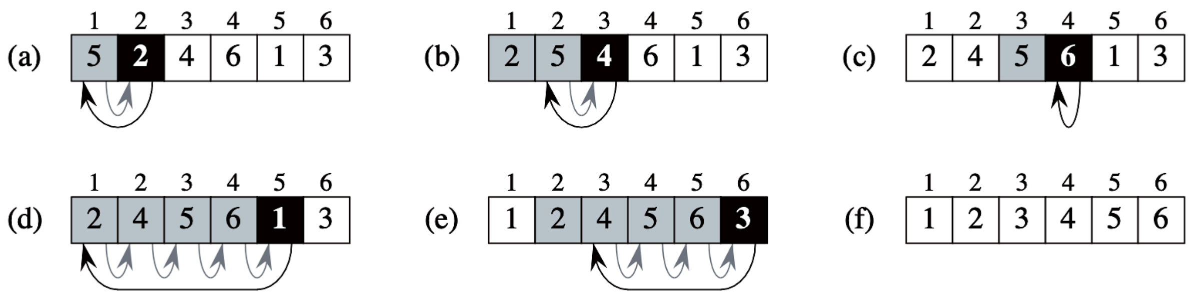
Less abstract than computational machines – but not deep in the weeds CMOS technology:

Mismatch cost for implementing pseudo-code

(Periodic process, but for simplicity, not local)

- Algorithm to sort any list of six integers into ascending order

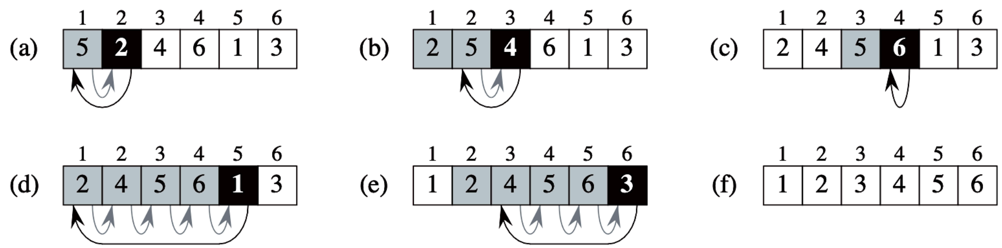
Insertion sort

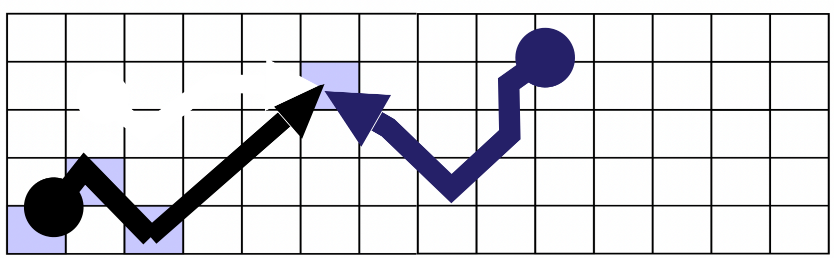


- All contributions to mismatch cost come from many-to-one maps over the sequence of six integers.

Just like many-to-one maps cause nonzero "Landauer cost", many-to-one maps cause nonzero mismatch cost

Insertion sort





- Each cell: different joint value of variables in the pseudo-code

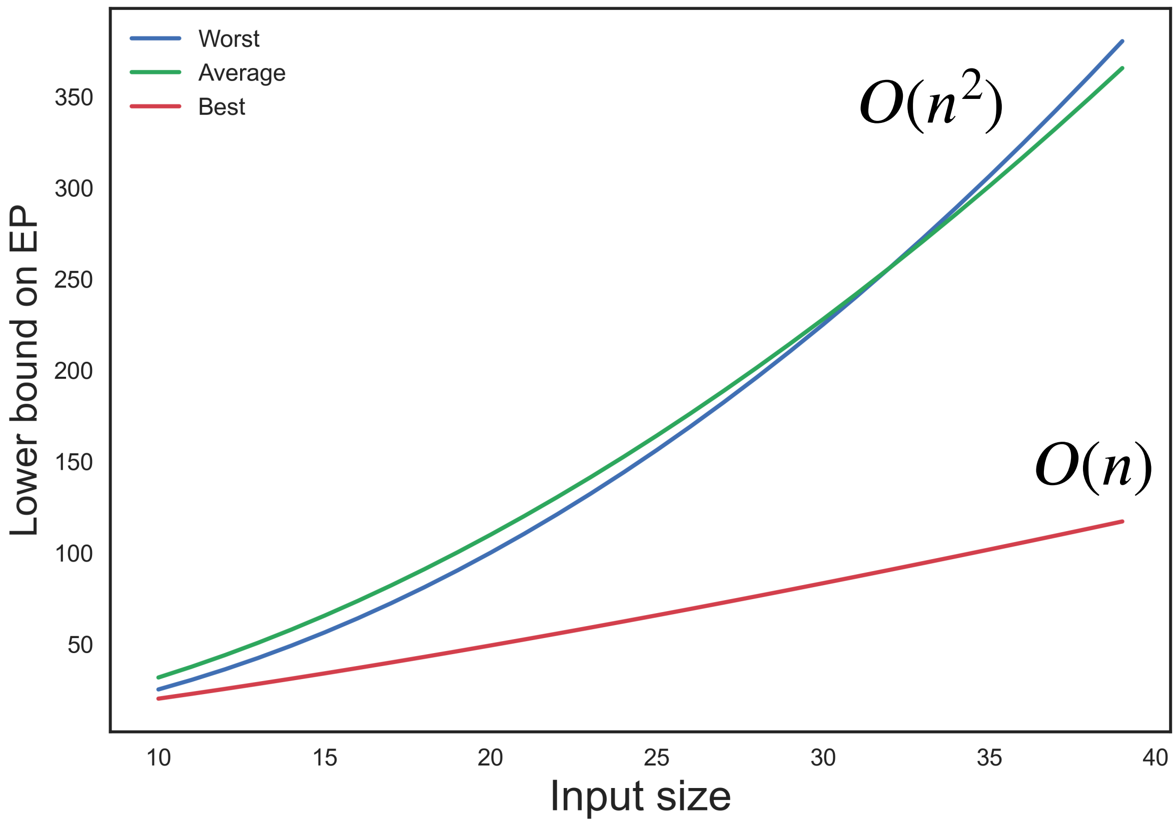
- Trajectories can merge
  - many-to-one maps

Total mismatch cost summed along a trajectory for a "maxent" (uniform) prior:

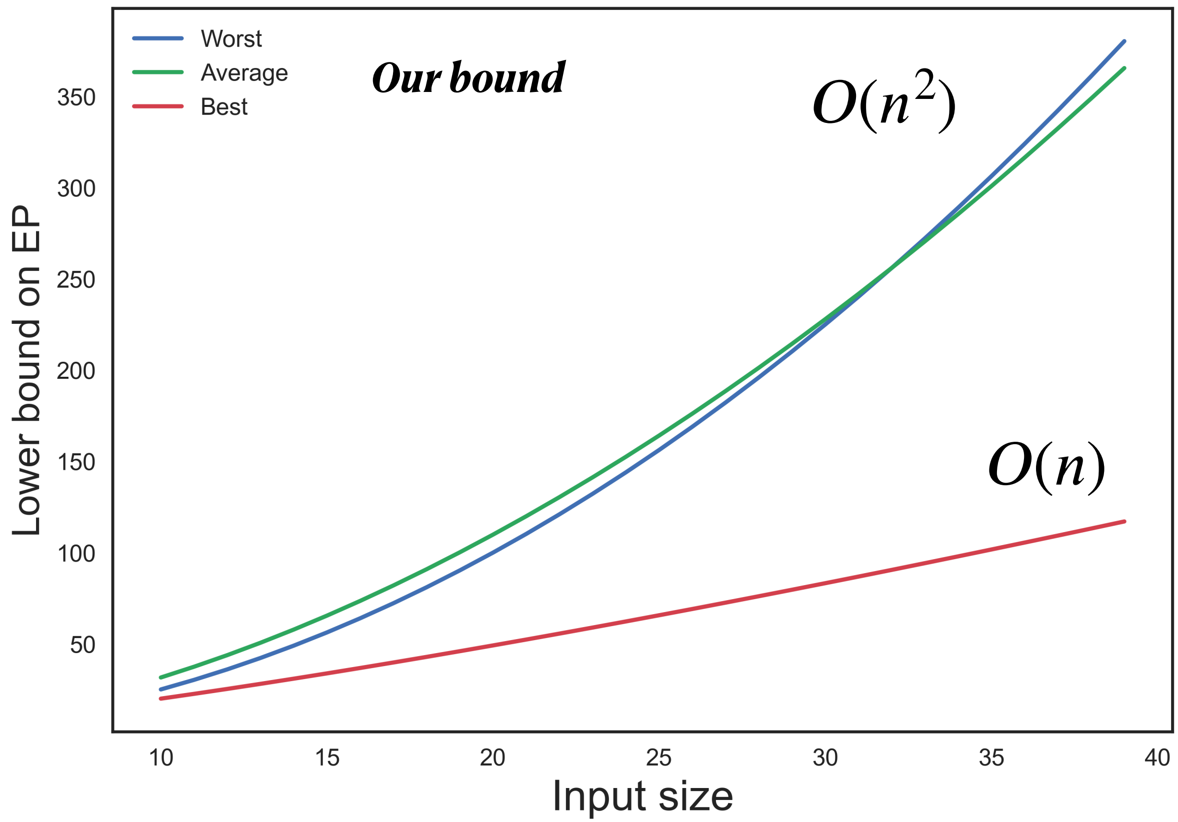
$$\mathscr{A}(\tau) = \ln\left(\epsilon_{X_1}\cdots\epsilon_{X_\tau}\right) = \sum_{s=1}^{\tau} \ln \epsilon_{X_s}$$

(where ϵ_{x_i} is "entrance rate" into state x_i, i.e., $\epsilon_{x_i} = \sum_{x_{i-1}} P(x_i|x_{i-1})$)

Energetic complexity!



Energetic complexity!



Landauer's bound

Time	Space (words)	Energy (bits)
$\Theta(n^2)$	$\Theta(n)$	0

Energy-Efficient Algorithms

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Kardes, G., Manzano, G.; Wolpert, D., Roldan, E. (2023)

Thermodynamics of Turing Machines

- There are many different abstract models of computers, with different computational powers.

- A particularly important one is the Turing machine (TM)

- Church-Turing thesis: *"Every function which would naturally be regarded as computable … is computable by a Turing machine." (Including computations in the human brain.)*

Turing Machines

1) Bi-infinite bit string ("tape") ***s***.

2) "***Head***" with n internal states y, one a "***halt state***"

3) At each t, head is located at bit b_t which has value $s(b_t)$.

4) At each t, based on (y_t, b_t), the head:

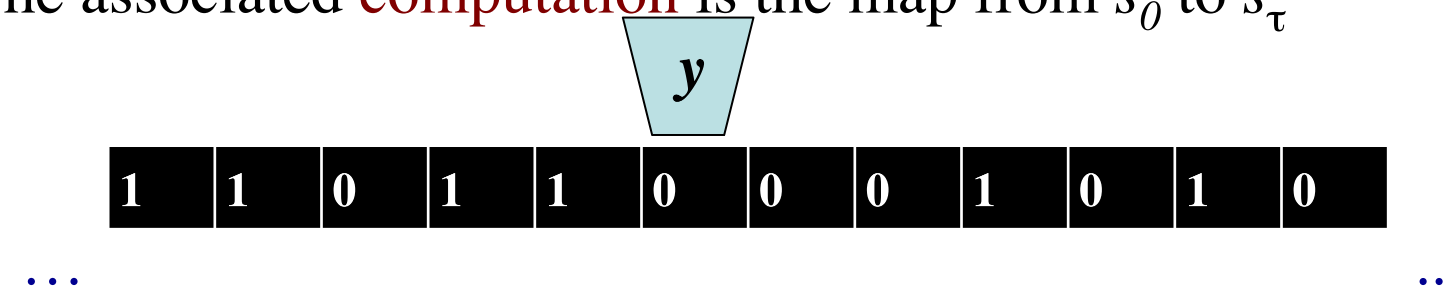
i) changes its state to y_{t+1};

ii) writes a new binary value at b_t;

iii) moves, by up to one bit, in either direction on the tape

5) Computation ends (if ever) at time τ if head halts then.

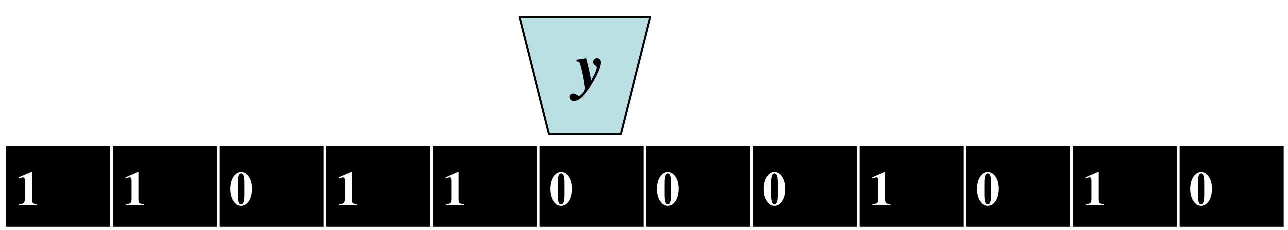
6) The associated computation is the map from s_0 to s_τ



Turing Machines

1) The standard model of computation (up to and including human "computers")

2) In particular, the Python interpreter on the laptop in front of you is a Turing machine.



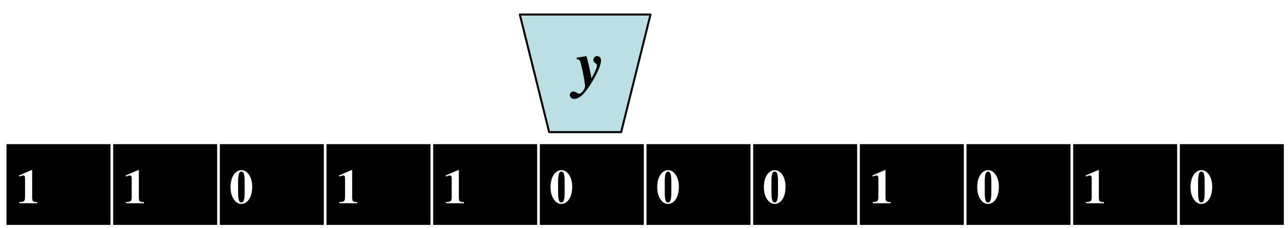
Turing Machines

1) The standard model of computation (up to and including human "computers")

2) Almost all binary-valued functions f(.) over bit strings cannot be computed by any TM.

Proof: Set of all TMs = $\{0, 1\}^*$, the set of all finite bit strings

Set of all f(.) = $2^{\{0, 1\}^*}$. **QED**.



Smallest input to a Turing Machine

Kolmogorov complexity of bit string v, $K(v)$:

Minimum number of non-zero bits in an input bit string that causes the Turing machine to produce output string v and halt.

- Very common (and powerful) measure of "how complex" v is.
- Related to Shannon entropy ("complexity" of a singleton rather than of a distribution)
- *"Uncomputable"*, i.e., no computer program can calculate it. (There is no Turing machine that takes any v as input and eventually produces $K(v)$ as output and halts.)

- Generate input strings ***s*** to a TM by ***coin-flipping*** distribution:

  $P(\mathbf{s}) = 2^{-|\mathbf{s}|} / Z$ (Normalization constant $Z \leq 1$)

- **Kolmogorov complexity** of bit string v, $K(v)$:

  Minimum *length* input string ***s*** to a given TM
  for it to compute v and halt.

- Bayes theorem:
  $K(v)$ is length of most probable input ***s***, given that output = v

Set thermo. rev. input distribution $q_0(\boldsymbol{s})$ to coin-flipping distribution

Thermodynamic complexity of bit string v:

Minimum *heat flow* for any input distribution $p_0(\boldsymbol{s})$ to a TM (that is reversible for $q_0(\boldsymbol{s})$) to compute v and then halt:

$$\boxed{K(v) + \log[G(v)] + \log[Z]}$$

where

- $K(v)$ is Kolmogorov complexity of v
- Z – the normalization constant – is Chaitin's constant
- $G(v)$ is probability of v under $q_0(\boldsymbol{s})$

Wolpert, D., *J. Phys. A* (2019)
Kolchinsky, A., Wolpert, D., *Phys. Rev. R* (2020)

Set thermo. rev. input distribution $q_0(\boldsymbol{s})$ to coin-flipping distribution

Thermodynamic complexity of bit string v:

Minimum *heat flow* for any input distribution $p_0(\boldsymbol{s})$ to a TM (that is reversible for $q_0(\boldsymbol{s})$) to compute v and then halt:

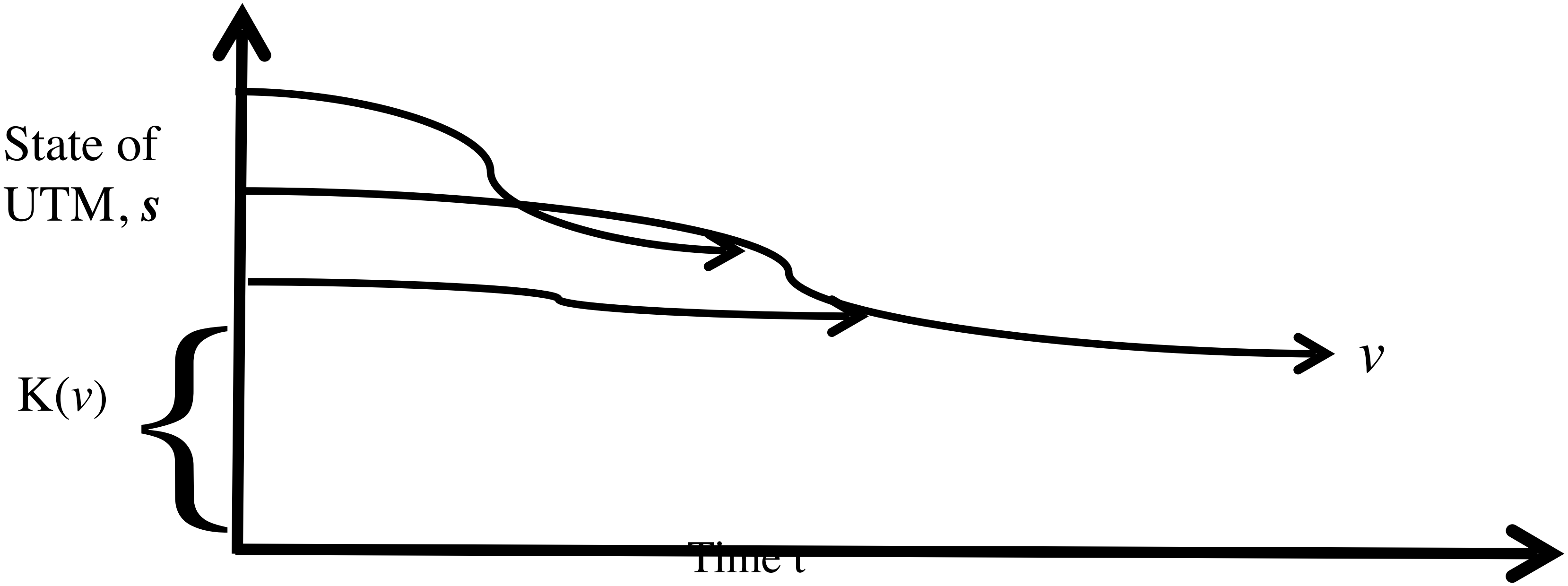
$$\boldsymbol{K(v) + log[G(v)] + log[Z]}$$

where

- $K(v)$ is Kolmogorov complexity of v
- Z – the normalization constant – is Chaitin's constant
- $G(v)$ is probability of v under $q_0(\boldsymbol{s})$

A "correction" to Kolmogorov complexity, reflecting ***cost of many-to-one maps as the TM evolves***

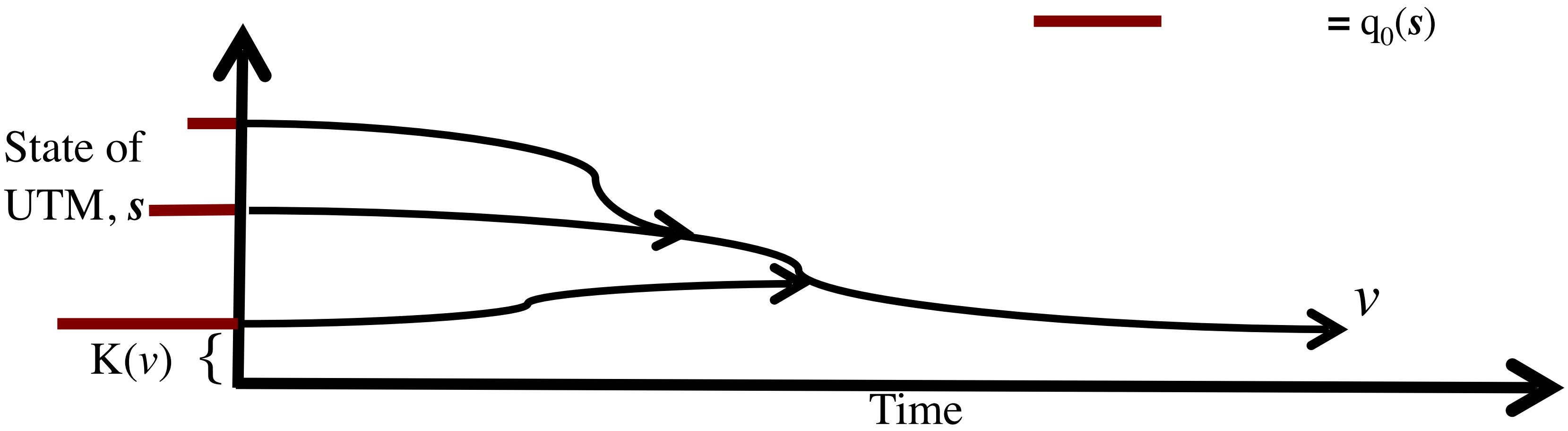
Kolmogorov complexity of *v*:



K(*v*) is ***unbounded*** **–** no constant exceeds length of {the shortest string to compute *v*} for all *v*

Thermodynamic complexity of *v*:

$$K(v) + \log[G(v)] + \log[Z]$$



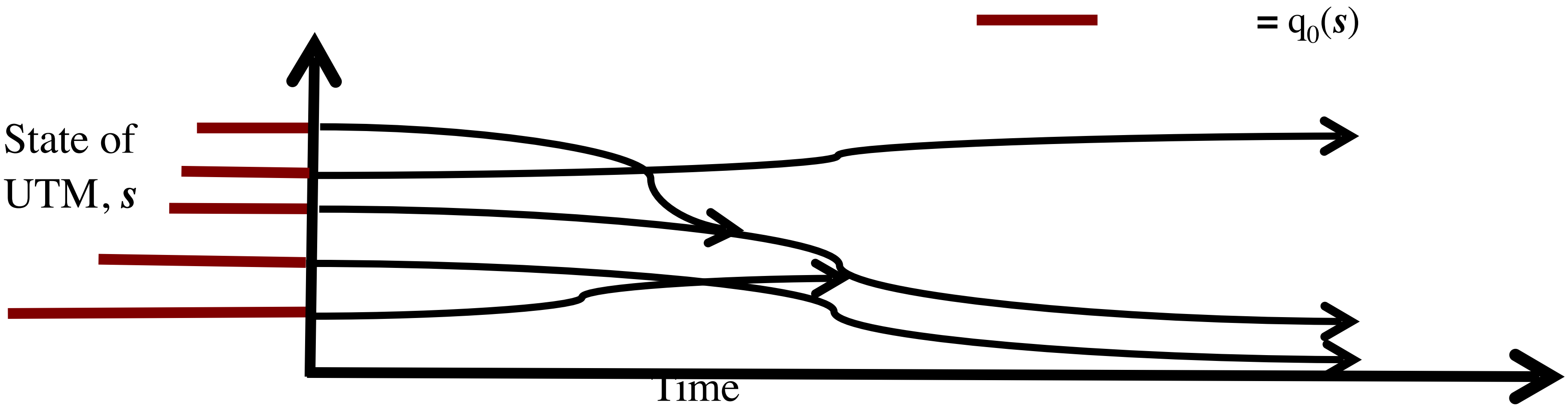
Minimal heat flow is ***bounded*** – there is a constant that exceeds {minimal EF to compute *v*} for all *v*:

log[sum of lengths of red lines]

Wolpert, D., J. Physics A (2019)

Thermodynamic complexity of *v*:

$$K(v) + \log[G(v)] + \log[Z]$$

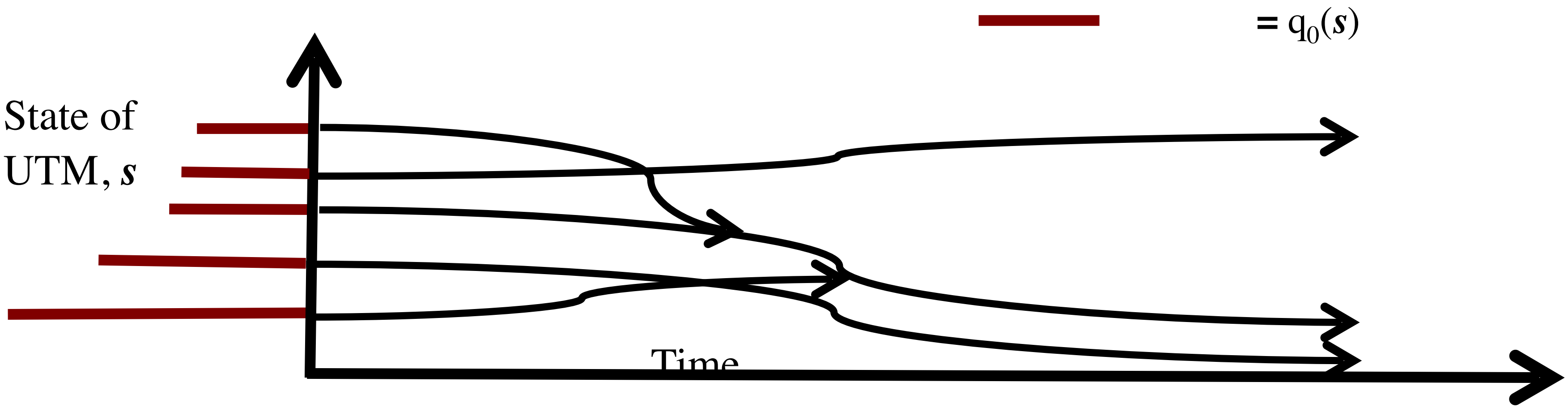


Expected heat flow is ***infinite***

Kolchinsky, A., and Wolpert, D.,Phys. Rev. Res. (2020)

Thermodynamic complexity of *v*:

$$K(v) \; + \; \log[G(v)] + \log[Z]$$



How do these results get modified if the update function of the TM is thermodynamically inefficient?

Answer: Who knows?

(See papers by Brittain et al., and Strasberg et al., in bibliography)

SUMMARY OF RESULTS

- Equation for heat expelled by a dynamic system (like a computer):

$$EF(p0) = \text{Landauer cost (p0)} + \text{EP(p0)}$$

- ***Now have broadly applicable bounds on the second resource cost, EP(p0)***

- Here consider two such bounds on that resource cost, **SLT** and **mismatch cost**.

1: Both SLT and mismatch cost distinguish among computationally equivalent circuits with identical number of gates

2: Rich behavior of mismatch cost for DFAs run for multiple iterations

3: Nontrivial scaling of mismatch cost with input size for insert-sort algorithm

4: Thermodynamic Kolmogorov complexity is ***bounded*** (unlike Kolmogorov complexity)

5: Average thermodynamic work to run a TM is ***infinite***

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